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COMPUTATION OF NON-STATIONARY WAVES ON THE SURFACE OF A HEAVY LIQUID OF FINITE DEPTH†

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A numerical scheme for computing non-stationary spatially periodic capillary-gravitational waves is presented. The use of an equation for the curvature of a curve changing with time obtained in the present paper makes it very different from the existing methods of boundary integral equations [1, 2]. The method is very accurate and efficient. Test results are presented along with computations of the collapse of a wave as the depth changes and jet formation for large-amplitude oscillations of a liquid.

1. THE BOUNDARY-VALUE PROBLEM FOR THE VELOCITY FIELD POTENTIAL

We introduce a Cartesian system of coordinates x, y with the vertical y axis pointing upwards. We can assume without loss of generality that the length of a wave period equals 2π and the acceleration due to gravity is equal to unity.

Let $x(t,s), y(t,s)$ be the parametric equation of the profile L of one wave period at time t, where s is the natural parameter ($ds^2 = dx^2 + dy^2$, $0 < s < l(t)$, $l(t)$ being the length of one wave period), and let the unperturbed wave surface and the bottom surface be defined by the equations $y=0$ and $y=-h$, respectively. It is always assumed that the normal vector is directed into the liquid.

This being the case, the velocity field potential Φ satisfies Laplace's equation in the domain Ω of the flow, as well as the following periodicity condition, the condition at the bottom, and the dynamic and kinematic conditions on the free surface L

$$\nabla^2\Phi = 0, \quad \Phi(x+2\pi, y) = \Phi(x,y), \quad \left.\frac{\partial\Phi}{\partial n}\right|_{y=-h} = 0 \tag{1.1}$$

$$\frac{\partial'\Phi}{\partial t} + \frac{1}{2}|\nabla\Phi|^2 + y - \sigma k = 0, \quad \frac{\partial\Phi}{\partial n} = v = \frac{\partial x}{\partial t}\frac{\partial y}{\partial s} - \frac{\partial y}{\partial t}\frac{\partial x}{\partial s} \text{ on } L \tag{1.2}$$

Here σ is the surface tension, k is the curvature, and v is the component of the velocity of liquid particles in the direction of the normal vector $\mathbf{n}$ directed into the liquid. By $\partial'/\partial t$ we understand the partial time derivative with the Cartesian coordinates x and y fixed.

Problem (1.1), (1.2) defines the velocity field potential Φ, provided that the motion of the profile of the capillary-gravitational wave on the surface of a liquid of finite depth is known.

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2. THE PARAMETRIC EQUATION OF THE FREE SURFACE

The profile of the free wave surface, which changes with time, will be described by the natural equation $k = k(t, s)$, where k is the curvature of the curve and s is the natural parameter (the arc length relative to a fixed point). then the radius vector $\mathbf{r}(x, y)$ and the x, y coordinates of the wave profile can be found by integrating the system of equations

$$\partial\theta/\partial s = k, \quad \partial \mathbf{r}/\partial s = \boldsymbol{\tau} \tag{2.1}$$

The unit vector τ parallel to the tangent line to the wave profile has components $\cos\theta$ and $\sin\theta$ where θ is the angle between the tangent line and the horizontal axis x.

It is convenient to express the natural parameter s in terms of a parameter z changing within fixed time-independent limits over one period

$$ds = l(t)f(z)dz, \quad 0 \leqslant z \leqslant 1 \quad \left(\int_0^1 f(z)dz = 1\right) \tag{2.2}$$

where l is the length of one wave period, so that $f(z)$ satisfies the above condition in parentheses.

It follows that the wave profile is determined by the following parametric equation: $\mathbf{r} = \mathbf{r}(z, t)$.

If points (markers) are placed on the wave profile at $z_i = i/N$ $(i = 1, 2, \ldots, N)$ with a constant step $\Delta z = 1/N$ in z, then the distance between them can be determined from (2.2) to be $\Delta s = lf(z_i)\Delta z_i$. It follows that $f(z)$ is the inverse of the density of markers.

We assume that $f(z)$ is independent of time. The ratio of the distances between the points will therefore be preserved in time. Such a distribution of markers was apparently first proposed in [2]. The later paper [3] involved the same distribution of markers and confirmed the stability of the numerical schemes.

3. EQUATION FOR THE CHANGE OF CURVATURE

Let $\upsilon(t, z)$ and $u(t, z)$ be the components of the velocity of markers, υ being the component normal to the wave profile and u the tangential component. Then

$$\partial\mathbf{r}/\partial t = \tau u - \nu\upsilon \tag{3.1}$$

where ν is the vector with components $-\sin\theta$, $\cos\theta$ perpendicular to τ. We express the change of curvature with time in terms of u and υ. To this end we change from s to z in (2.1) with the aid of (2.2)

$$\partial\mathbf{r}/\partial z = \tau l f \tag{3.2}$$

It follows from (3.1) and (3.2) that

$$\partial^2\mathbf{r}/\partial t\partial z = \partial(\tau u - \nu\upsilon)/\partial z = \partial(\tau l f)/\partial t = \partial^2\mathbf{r}/\partial z\partial t \tag{3.3}$$

We use the formulae

$$\frac{\partial\tau}{\partial z} = \nu\frac{\partial\theta}{\partial z}, \quad \frac{\partial\nu}{\partial z} = -\tau\frac{\partial\theta}{\partial z}, \quad \frac{\partial\tau}{\partial t} = \nu\frac{\partial\theta}{\partial t}$$

for the derivatives of τ and ν with respect to z and t and substitute them into (3.3). We then obtain a vector equation, which implies the two scalar equations

$$\frac{\partial u}{\partial z} + \upsilon\frac{\partial\theta}{\partial z} = f\frac{dl}{dt}, \quad \frac{\partial\theta}{\partial t} = \frac{u}{lf}\frac{\partial\theta}{\partial z} - \frac{1}{lf}\frac{\partial\upsilon}{\partial z} \tag{3.4}$$

Differentiating the second equation in (3.4) with respect to z and setting

$$K = kl, \quad V = v/l, \quad U = u/l, \quad \partial\theta/\partial z = fkl = fK, \tag{3.5}$$

we obtain the desired equation

$$\frac{\partial fK}{\partial t} = \frac{\partial}{\partial z}\,\frac{UfK - \partial V/\partial z}{f} \tag{3.6}$$

for the change in the curvature of the profile. Equation (3.6) has the divergent form, suitable for constructing a numerical scheme conservative with respect to fK.

4. EQUATION FOR THE ELEMENT OF LENGTH AND TANGENTIAL VELOCITY

Using the substitution (3.5), the first equation in (3.4) can be represented in the form

$$\frac{\partial U}{\partial z} = \frac{f(z)}{l}\,\frac{dl}{dt} - fKV(z) \tag{4.1}$$

Integrating equation (4.1) with respect to z from 0 to 1 and using the periodicity of $U(z)$ and the condition in parentheses in (2.2), we get

$$\frac{1}{l}\,\frac{\partial l}{dt} = \int_0^1 fKV(z')dz' \tag{4.2}$$

Integrating (4.1) with respect to z using (4.2), we obtain the expression

$$U = U_0 + \int_0^z \left(f(z')\int_0^1 fKV(z'')dz'' - fKV(z')\right) dz' \quad \left(\int_0^1 Udz = 0\right) \tag{4.3}$$

for the tangential velocity. The arbitrary constant U_0 should be computed from the condition in parentheses.

The equation

$$\frac{\partial}{\partial t}ds = \left(kv + \frac{\partial u}{\partial s}\right) ds \tag{4.4}$$

for the variation of the arc length element ds (the distance between markers) in time is interesting. The first term defines the variation of ds due to the motion of markers with velocity v in the normal direction. The second term defines the increment of ds due to the motion of markers with various velocities u in the tangential direction to the curve.

At the same time, (2.2) implies that the variation of the distance between markers is proportional to the length of the profile

$$\partial(ds)/\partial t = (l^{-1}dl/dt)ds \tag{4.5}$$

The equation following from (4.4) and (4.5) is equivalent to Eq. (3.4) already obtained, which defines the velocity of motion of the markers in the tangential direction to the curve.

5. EQUATION FOR THE POTENTIAL ON THE FREE SURFACE

Using (2.2) and (3.5), we express the partial derivative $\partial'/\partial t$ with x and y fixed in terms of the derivative $\partial/\partial t$ for constant z

$$\frac{\partial'\Phi}{\partial t} = \frac{\partial\Phi}{\partial t} - u\frac{\partial\Phi}{\partial s} - v\frac{\partial\Phi}{\partial n} = \frac{\partial\Phi}{\partial t} - \frac{U}{f}\frac{\partial\Phi}{\partial z} - V^2 l^2$$

Then the dynamic boundary condition (1.2) on the free surface can be represented as

$$\frac{\partial\Phi}{\partial t} = \frac{1}{2}V^2 l^2 - y + U\left(\frac{1}{f}\frac{\partial\Phi}{\partial z}\right) - \frac{1}{2l^2}\left(\frac{1}{f}\frac{\partial\Phi}{\partial z}\right)^2 + \frac{\sigma f K}{lf} \tag{5.1}$$

This equation defines the potential on the wave profile as a function $\Phi(t, z)$. To close the system of equations it is necessary to obtain an expression for the velocity $V(t, z)$.

6. EVALUATION OF THE NORMAL VELOCITY FROM THE POTENTIAL ON THE FREE SURFACE

The stream function Ψ is a harmonic function in the domain of the flow of the liquid and satisfies the condition $\psi(x, -h) = \Psi_h = \text{const}$ at the bottom. It can be shown that the values $\Psi(Q)$ and $\Phi(Q)$ of the stream function and the potential on the free surface are connected by the linear relation

$$-\int_L \left(W(Q, Q')\frac{\partial\Phi}{\partial s'}(Q') + (\Psi(Q') - \Psi(Q))\frac{\partial W}{\partial n'}(Q')\right)ds' = \pi(\Psi(Q) - \Psi_h) \tag{6.1}$$

where $W(Q, Q')$ is Green's function for the Dirichlet problem in the domain of the flow. Integration is carried out over the wave profile L, $Q'(x', y')$ is the integration point belonging to L, ds' is the element of the arc at Q', and $Q(x, y)$ is a fixed point. The function $W(Q, Q')$ has the form

$$W(Q, Q') = W(x, y, x', y') = \frac{1}{2}\ln\frac{\operatorname{ch}\bar{y} - \cos\bar{x}}{1 - 2E\cos\bar{x} + E^2}, \quad E = e^{-(y+y'+2h)} \tag{6.2}$$

$$\bar{x} = x' - x, \quad \bar{y} = \bar{y}' - y \tag{6.3}$$

7. NUMERICAL DIFFERENTIATION AND INTEGRATION FORMULAE

Let $F(z)$ be a function of period 1 represented by a cubic spline

$$P_i(z) = F_{i-1}q_1 + F_i q_2 + \alpha_{i-1}q_3 + \alpha_i q_4 \tag{7.1}$$

in the interval $z_{i-1} < z < z_i$, $z_i = i/N$ $(i = 1, 2, \ldots, N)$, where F_i are the values of $F(z)$ at $z = z_i$ and q_i are the following cubic polynomials of $X = N(z - z_{i-1})$, where $0 \leqslant X < 1$

$$q_1 = 1 - X, \quad q_2 = X, \quad q_3 = -(X^3 - 3X^2 + 2X)/6, \quad q_4 = (X^3 - X)/6$$

The values a_i can be found by the pivotal condensation method from the conditions of continuity for the first and second derivatives of the spline at the mesh nodes $z = z_i$ and the periodicity condition.

The differentiation and integration formulae can be expressed in terms of the coefficients of a_i

$$\left.\frac{dF}{dz}\right|_{z_i} = \frac{N}{2}\left(F_{i+1} - F_{i-1} - \frac{1}{12}(\alpha_{i+1} - \alpha_{i-1})\right) \tag{7.2}$$

$$\int_{z_{i-1}}^{z_i} F(z)dz = \frac{1}{2N}(F_{i+1} + F_{i-1} - \frac{1}{24}(\alpha_{i+1} + \alpha_{i-1})) \tag{7.3}$$

Formula (7.2) is accurate to within N^{-3} and (7.3) is accurate to within N^{-4}.

The formula for integrating over one period that follows from (7.3) has the form

$$\int_0^1 F(z)dz = \frac{1}{N}\sum_{i=1}^{N} F_i + R_N, \quad R_N = \frac{1}{720N^4}F^{(IV)}(\xi), \quad 0 < \xi < 1 \tag{7.4}$$

The coefficient of the remainder R_N in (7.4) is eight times smaller compared with the remainder in Simpson's formula.

To integrate periodic functions with a logarithmic singularity one can obtain the following quadratic formula for an even number $N = 2M$ of markers

$$\int_0^1 \ln|\sin(z - z_i)\pi| F(z)dz = \sum_{j=1}^{N} \alpha(i-j)F_j$$
$$\alpha(m) = \frac{-1}{N}(\ln 2 + \sum_{j=1}^{M-1}\frac{1}{j}\cos(2\pi j\frac{m}{N})) + \frac{(-1)^m}{N} \tag{7.5}$$

Formula (7.5) is accurate for all trigonometric polynomials of order N.

8. APPROXIMATION OF THE INTEGRAL EQUATION FOR THE STREAM FUNCTION

Using the quadrature formulae (7.4) and (7.5), one can obtain the following approximation of Eq. (6.1)

$$\sum_{j=1}^{N}(A_{ij}\frac{\partial\Phi}{\partial z_j} + B_{ij}\Psi_j) = \pi(\Psi_i - \Psi_h) \tag{8.1}$$
$$A_{ij} = -\frac{1}{N}(W(x_i, y_i, x_j, y_j) + \beta|i - j|), \; i \neq j$$
$$\beta(m) = -\ln|\sin\pi\frac{m}{N}| + N\alpha(m), \quad m > 1$$

$$A_{ii} = -\frac{1}{N}(\frac{1}{2}\ln\frac{l^2 f^2(z_i)}{2\pi^2(1 - e^{-2(y_i+h)})^2}) - \alpha(0) \tag{8.2}$$

$$B_{ij} = -\frac{1}{N}(\frac{\partial W}{\partial x_j}\sin\theta_j - \frac{\partial W}{\partial y_j}\cos\theta_j), \; i \neq j, \quad B_{ii} = -\sum_{k\neq i} B_{ik} \tag{8.3}$$

The matrix coefficients B_{ij} are fairly small on the free surface, and they are identically equal to zero on the free surface of the form $y = 0$. Thus, iterative methods can be used to determine Ψ from (8.1). One or two iterations are usually sufficient to compute non-stationary waves.

The component V of the velocity normal to the wave profile can be determined by differentiating

$$V = (l^2 f)^{-1}\partial\Psi/\partial z \tag{8.4}$$

The accuracy of the computation of the stream function Ψ and the rate of convergence of the method can be verified by the following calculations. From the exact solution $\Psi = \sin(x)\mathrm{sh}(y + h)$ of the Laplace equation on the curve $x = 2\pi z$, $y = \sin(2\pi z)$ we compute the exact values $\partial\Phi'/\partial z = -f(z)\partial\Psi/\partial n$ and substitute them into (8.1). The approximate values Ψ_i obtained by solving the system of equations (8.1) are compared with the exact values of the stream function $\Psi_{ex} = \sin(2\pi z_i)\mathrm{sh}(\sin(2\pi z_i) + h)$.

As a function of N, the relative error $\epsilon(N)=\max|\Psi-\Psi_{ex}|/\max|\Psi|$ decreases for $h=100$ as follows: $\epsilon(8)=10^{-3}$, $\epsilon(12)=7\times10^{-6}$, $\epsilon(16)=3\times10^{-7}$. For $h=1.5$ we have $\epsilon(8)=10^{-3}$, $\epsilon(12)=2\times10^{-5}$, $\epsilon(16)=9\times10^{-7}$.

9. THE COMPLETE SYSTEM OF EQUATIONS

Let us state the final system of equations describing the time evolution of the wave

$$\sum_{j=1}^{N}\left(A_{ij}\frac{\partial\Phi}{\partial z_j}+B_{ij}\Psi_j\right)=\pi\Psi_i$$

$$V=(l^2 f(z))^{-1}\partial\Psi/\partial z$$

$$U=U_0+\int_0^z\left(f(z')\int_0^1 fKV(z'')dz''-fKV(z')\right)dz'$$

$$\frac{\partial fK}{\partial t}=\frac{\partial}{\partial z}\,\frac{UfK-\partial V/\partial z}{f} \qquad (9.1)$$

$$\frac{\partial\Phi}{\partial t}=\frac{1}{2}V^2l^2-y+U\left(\frac{1}{f}\frac{\partial\Phi}{\partial z}\right)-\frac{1}{2l^2}\left(\frac{1}{f}\frac{\partial\Phi}{\partial z}\right)^2+\frac{\sigma fK}{lf}$$

$$\theta=\theta_0+\int_0^z fk\,dz',\qquad x=x_0+l\int_0^z f\cos\theta\,dz',\qquad y=y_0+l\int_0^z f\sin\theta\,dz'$$

The matrices A_{ij} and B_{ij} can be computed from (8.2) and (8.3). The integration constants U_0, θ_0, x_0, y_0 and l can be found from the following requirements. The periodicity conditions must be satisfied exactly: $U(z+1)=U(z)$, $\theta(z+1)=\theta(z)$, $y(z+1)=y(z)$, and $x(z+1)=x(z)+2\pi$; the functions $U(z)$ and $y(z)$ must satisfy the conditions

$$\int_0^1 U(z)dz=0,\qquad \int_0^1 y(z)dx(z)=0$$

The fourth and fifth equations in (9.1) can be approximated by a system of differential equations of order $2N$ and can be solved by the Adams method of order four. At the initial instant the values k_i and Φ_i of the curvature and potential are given at N points lying on the wave surface. The distances between the markers are defined by the function $f(z)$. It is advisable to define $f(z)$ in such a way that the maximum point lies in the vicinity of the largest curvature of the wave profile.

10. FORMULAE FOR CHANGING TO A SYSTEM OF COORDINATES MOVING AT CONSTANT VELOCITY c

When computing travelling non-stationary waves it proves convenient to change to a moving system of coordinates. To this end it suffices to substitute

$$V=V'+c(l^2f)^{-1}\partial y/\partial z,\qquad U=U'+c(l^2f)^{-1}\partial x/\partial z \qquad (10.1)$$

into Eq. (5.1). The parameter c is convenient for computing breaking waves. A suitable value of c can be chosen by requiring that the minimum of $f(z)$ should lie at the point of maximum curvature of the wave profile. The highest density of markers will then be found in the neighbourhood of that point.

11. TESTING THE NUMERICAL METHOD OF COMPUTING NON-STATIONARY WAVES

The accuracy of the method can be checked for periodic standing waves, for which we [4] obtained solutions expressed as expansions with respect to a small parameter, namely, the save amplitude a. The solutions were obtained using the REDUCE analytic computation system [5]. Analogous expansions were presented in [6, 7]. One can represent the total wave energy E, the potential energy $E_p(\omega t)$ and the equation for the wave profile $y(\omega t, x)$ in this way, where ω is the angular frequency of oscillations. In particular, at the instants of time $\omega t_1 = \pi/2$ and $\omega t_2 = \pi$ we obtain

$$
\begin{aligned}
E_p(\pi/2) &= E = \frac{\pi a^2}{2}\left(1 - \frac{1}{2}a^2 + \frac{223}{616}a^4 - 0.10261a^6 + \dots\right) \\
E_p(0) &= E_p(\pi) = \frac{\pi a^2}{2}(0.02084a^6 + \dots) \\
y(0,\pi) &= y(\pi,\pi) = \frac{41}{336}a^4 - 0.12983a^6 + \dots \\
y(\pi/2,\pi) &= a + \frac{1}{2}a^2 - \frac{19}{112}a^4 + 0.04122a^6 + \dots \\
\omega &= 1 - \frac{1}{8}a^2 + \frac{11}{256}a^4 - 0.004492a^6 + \dots
\end{aligned}
\qquad (11.1)
$$

to within a^8.

To estimate the accuracy and convergence of the numerical method, computations of a standing wave were made for amplitudes $a = 0.2$ and $a = 0.3$. (For $a = 0.3$ the order of the remainders in (11.1) is approximately 10^{-5}.) In Table 1 the numerical results are compared with the values of the corresponding quantities obtained from (11.1). Since the kinetic energy is equal to zero for the most-developed wave, $E(\pi/2) = E_p(\pi/2)$.

12. EXAMPLES OF THE COMPUTATIONS OF WAVES

Computations for the problem of a breaking wave in deep water when the depth changes to $h = 1$ and for the problem of forming a cumulative stream in the case of periodic standing oscillations of a liquid in a tank have been carried out.

The collapse of a wave was computed as follows. The profile of a progressive wave of given finite amplitude in a heavy liquid of given depth was computed using the scheme presented in [8].

TABLE 1

Quantity	Computation based on formulae (11.1)		Error of the numerical solution			
			$a = 0.2$		$a = 0.3$	
	$a = 0.2$	$a = 0.3$	$N = 8$	$N = 32$	$N = 8$	$N = 32$
$E_p(\pi/2)$	0.061612	0.135414	7×10^{-4}	2×10^{-6}	1×10^{-4}	5×10^{-6}
$y(\pi/2, \pi)$	0.219711	0.343656	6×10^{-3}	2×10^{-5}	1×10^{-3}	4×10^{-5}
$E(\pi)$	0.061611	0.135414	3×10^{-4}	9×10^{-7}	4×10^{-5}	1×10^{-6}
$E_p(\pi)$	8×10^{-8}	215×10^{-8}	7×10^{-5}	1×10^{-8}	4×10^{-6}	2×10^{-8}
$y(\pi, \pi)$	1.9×10^{-4}	8.9×10^{-4}	1×10^{-2}	2×10^{-5}	3×10^{-3}	7×10^{-5}

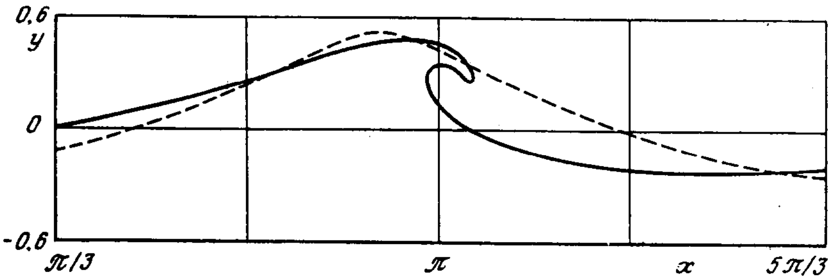

FIG. 1.

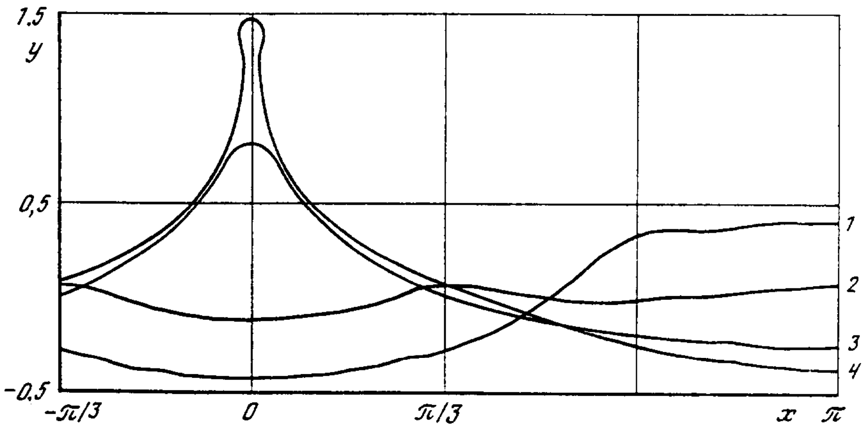

FIG. 2.

All the resulting wave characteristics (velocity of propagation, energy and momentum) were compared with the corresponding results in [9] obtained by the series summation method. The results were consistent to within all significant digits given in [9]. The resulting profile and the velocity field were then taken as the initial data, and the variation of the wave profile with time was computed for a liquid of different depth. The computation was carried out using the method proposed in the present paper.

In Fig. 1 the dashed line represents the profile of the progressive wave with amplitude $a = 0.406$ in a liquid of infinite depth corresponding to [9]. The solid line represents the profile of a breaking wave when the depth changes to $h = 1$.

Similar breaking waves in deep water subject to a variable wind load were considered in [10].

In the second example a stationary sinusoidal profile of amplitude $a = 0.5$ is given at $t = 0$. The depth is infinite. In the linear approximation this initial condition corresponds to a standing periodic wave. Because of non-linearity, a cumulative stream is formed after several periods. the free surfaces at times $t = 13.5$; 14.5; 15/5; 16.3 are shown in Fig. 2 (curves 1–4 are symmetrical about the axis $x = 0$).

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